

# Absence of solid angle deficit singularities in beyond-generalized Proca theories

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In Gleyzes-Langlois-Piazza-Vernizzi (GLPV) scalar-tensor theories, which are outside the domain of second-order Horndeski theories, it is known that there exists a solid angle deficit singularity in the case where the parameter  $\alpha_H$  characterizing the deviation from Horndeski theories approaches a non-vanishing constant at the center of a spherically symmetric body. Meanwhile, it was recently shown that second-order generalized Proca theories with a massive vector field  $A^\mu$  can be consistently extended to beyond-generalized Proca theories, which recover shift-symmetric GLPV theories in the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$ . In beyond-generalized Proca theories up to quartic-order Lagrangians, we show that solid angle deficit singularities are generally absent due to the existence of a temporal vector component. We also derive the vector-field profiles around a compact object and show that the success of the Vainshtein mechanism operated by vector Galileons is not prevented by new interactions in beyond generalized Proca theories.

## I. INTRODUCTION

The constantly accumulating observational evidence for an acceleration of the Universe today [1–3] implies that we may require additional dynamical degrees of freedom (DOF) to those appearing in standard model of particle physics. The vacuum energy arising in standard quantum field theory can be responsible for the cosmic acceleration, but the problem is that the predicted energy scale is enormously larger than the observed dark energy scale [4]. Instead of resorting to the vacuum energy, there have been numerous theoretical attempts for constructing dark energy models with some new dynamical DOF [5].

A minimally coupled scalar field with a potential is one of the simplest dynamical dark energy models [6]. As in Brans-Dicke theory [7], the scalar field can also be non-minimally coupled to the Ricci scalar. Moreover, we can also think of derivative interactions of the scalar field with the Ricci scalar and the Einstein tensor. Covariant Galileons [8, 9] have such derivative interactions with gravity (see Ref. [10] for Minkowski Galileons). Modified gravitational theories were usually constructed to keep the equations of motion up to second order for avoiding the ghost-like Ostrogradski instability [11]. Most general scalar-tensor theories with second-order equations of motion are known as Horndeski theories [12, 13], which accommodate a wide range of single scalar dark energy models proposed in the literature.

Gleyzes-Langlois-Piazza-Vernizzi (GLPV) [14] performed a healthy extension of Horndeski theories after expressing the Horndeski Lagrangian in terms of ADM scalar quantities with the choice of unitary gauge [15]. Even if the equations of motion can be higher than second order in such generalizations, it was shown that the number of dynamical scalar DOF remains one in GLPV theories [16]. In the cosmological set-up, the deviation from Horndeski theories gives rise to the mixing between sound speeds of the scalar field and the matter sector [14, 17, 18]. This property provides tight constraints on some dark energy models beyond the Horndeski domain [19, 20].

In GLPV theories, it was shown in Refs. [21, 22] that a solid angle deficit singularity appears in the case where the parameter  $\alpha_H$  characterizing the departure from Horndeski theories approaches a non-zero constant at the center of a spherically symmetric body. In this case, the Ricci scalar has the dependence  $R = -2\alpha_H/r^2$  around the center of body ( $r = 0$ ), so  $R$  exhibits the divergence at  $r = 0$ . This problem is also related to the breaking of the Vainshtein mechanism [23] at small radius found in Ref. [24] (see also Refs. [25]). To avoid these problems, the viable models need to be constructed in such a way that  $\alpha_H$  vanishes for  $r \rightarrow 0$  [21, 26]. In such non-singular cases the Vainshtein mechanism can be at work inside the solar system, while realizing the successful cosmic expansion history [27].

Scalar-tensor theories are not the only possibility for the construction of viable dark energy models, but a vector field can be also the source for the late-time cosmic acceleration [28–35]. For a massive vector-field theory (Proca theory), the  $U(1)$  gauge symmetry is explicitly broken, so that the longitudinal mode propagates. For the construction of theoretically consistent theories, the crucial requirement is that self-interactions of the longitudinal mode belong to those of Galileon/Horndeski theories. The condition of second-order equations of motion on curved backgrounds enforces the presence of non-minimal derivative couplings with gravity. An interesting subclass of these type of vector-tensor theories naturally arises in modified gravity theories with Weyl geometries [30]. A systematical classification of derivative vector self-interactions with three propagating degrees of freedom was carried out in Ref. [36]. The correct number of physical degrees of freedom is guaranteed by the propagation of second class constraint. Its presence was checked by computing the Hessian matrix and the number of its vanishing eigenvalues. These theories were further

investigated in Refs. [37, 38]. In such theories, there are two transverse vector modes and one longitudinal scalar besides two tensor polarizations. Taking the limit  $A^\mu \rightarrow \nabla^\mu \chi$ , where  $\chi$  is a scalar field, the action of generalized Proca theories recovers the shift-symmetric action of Horndeski theories [36]. The dark energy cosmology and spherically symmetric solutions in generalized Proca theories were extensively studied in Refs. [39–43].

In Ref. [44] the present authors extended second-order generalized Proca theories in such a way that the action of new theories can reproduce shift-symmetric GLPV theories by taking the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$ . On the isotropic and anisotropic cosmological backgrounds, it was shown that beyond-generalized Proca theories do not give rise to additional DOF associated with the Ostrogradski ghost to those arising in generalized Proca theories [44, 45]. The absence of extra ghostly DOF up to quartic-order beyond-generalized Proca theories was also confirmed in Ref. [46] on general curved backgrounds. Hence it is possible to construct healthy massive vector theories even outside the domain of second-order generalized Proca theories.

Given the fact that GLPV theories can have the solid angle deficit singularity at the center of a compact object, we are interested in what happens for beyond-generalized Proca theories in the spherically symmetric setup. In GLPV theories the source of solid angle deficit singularities is related to the geometric modification of the quartic Horndeski Lagrangian. In this paper, we consider the quartic-order beyond-generalized Proca Lagrangians (whose explicit forms are given in Sec. II) and show in Sec. IV that solid angle deficit singularities do not generally appear due to the existence of the temporal vector component (see Sec. III for the comparison with GLPV theories). Thus, beyond-generalized Proca theories have an advantage over GLPV theories in that it is not necessary to design models for avoiding the problem of solid angle deficit singularities. In Sec. V, we also study how the new Lagrangian affects the vector-field profile around the compact object and show that it does not prevent the success of the Vainshtein mechanism operated by vector-Galileon terms. We conclude in Sec. VI.

## II. QUARTIC-ORDER BEYOND-GENERALIZED PROCA THEORIES AND THE SPHERICAL SYMMETRIC SETUP

The problem of solid angle deficit singularities in GLPV theories can arise in the presence of the quartic Lagrangian  $L_4$  beyond the domain of Horndeski theories [21]. The quintic Lagrangian  $L_5$  of GLPV theories does not modify the property of singularities generated by  $L_4$  [22]. Hence the Lagrangian  $L_4$  is crucial for the existence of solid angle deficit singularities. In this paper, we shall study spherically symmetric solutions in beyond-generalized Proca theories up to quartic order. Besides the vector field  $A^\mu$  coupled to the Ricci scalar  $R$ , we take into account the matter Lagrangian density  $\mathcal{L}_m$ . The action of quartic-order beyond-generalized Proca theories is given by [44]

$$S = \int d^4x \sqrt{-g} \left( \mathcal{L}_F + \sum_{i=2}^4 \mathcal{L}_i + \mathcal{L}_4^N + \mathcal{L}_m \right), \quad (2.1)$$

where  $g$  is a determinant of the space-time metric  $g_{\mu\nu}$ , and

$$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.2)$$

$$\mathcal{L}_2 = G_2(X), \quad (2.3)$$

$$\mathcal{L}_3 = G_3(X) \nabla_\mu A^\mu, \quad (2.4)$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) [(\nabla_\mu A^\mu)^2 - \nabla_\rho A_\sigma \nabla^\sigma A^\rho] + \frac{1}{2} g_4(X) F_{\mu\nu} F^{\mu\nu}, \quad (2.5)$$

$$\mathcal{L}_4^N = f_4(X) \mathcal{E}_{\alpha_1 \alpha_2 \alpha_3 \gamma_4} \mathcal{E}^{\beta_1 \beta_2 \beta_3 \gamma_4} A^{\alpha_1} A_{\beta_1} \nabla^{\alpha_2} A_{\beta_2} \nabla^{\alpha_3} A_{\beta_3}, \quad (2.6)$$

with  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ , and  $\nabla_\mu$  being the covariant derivative operator. The functions  $G_{2,3,4}, g_4, f_4$  depend on the quantity

$$X \equiv -\frac{1}{2} A_\mu A^\mu. \quad (2.7)$$

For the partial derivative with respect to  $X$ , we use the notation  $G_{i,X} \equiv \partial G_i / \partial X$ . In  $\mathcal{L}_4$  the non-minimal derivative coupling  $G_4(X) R$  is required to keep the equations of motion up to second order. The last term of Eq. (2.5) corresponds to the intrinsic vector mode, which vanishes by taking the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$  [36, 37]. The  $U(1)$  gauge invariance is explicitly broken by introducing the massive Proca Lagrangian  $m^2 X$  in  $\mathcal{L}_2$ , in which case the longitudinal mode of the vector field propagates. The number of propagating DOF in second-order generalized Proca theories is five on general curved backgrounds (one longitudinal scalar, two transverse vector modes, and two tensor polarizations).

The Lagrangian density  $\mathcal{L}_4^N$ , which contains products of the Levi-Civita tensor  $\mathcal{E}_{\alpha_1\alpha_2\alpha_3\gamma_4}$  and the covariant derivatives of  $A_\mu$  up to first order, was constructed in a manner analogous to the GLPV extension of scalar-tensor Horndeski theories. Taking the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$ ,  $\mathcal{L}_4^N$  recovers the quartic Lagrangian density of GLPV theories with the function  $f_4$  depending on  $X = -\nabla_\mu \chi \nabla^\mu \chi / 2$  alone. Although the new interaction  $\mathcal{L}_4^N$  is outside the domain of second-order generalized Proca theories, it does not give rise to additional ghostly scalar DOF on the isotropic cosmological background [44], on the Bianchi type I background [45], and on general backgrounds [46].

To study problems of solid angle deficit singularities and the Vainshtein screening inside and outside a compact object, we consider a static and spherically symmetric space-time described by the line element

$$ds^2 = -e^{2\Psi(r)} dt^2 + e^{2\Phi(r)} dr^2 + r^2 d\Omega^2, \quad (2.8)$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ , and the gravitational potentials  $\Psi(r)$  and  $\Phi(r)$  depend on the distance  $r$  from the center of symmetry. We deal with the matter Lagrangian density  $\mathcal{L}_m$  as a perfect fluid characterized by the energy-momentum tensor  $T_\nu^\mu = \text{diag}(-\rho_m, P_m, P_m, P_m)$ . We assume that matter is minimally coupled to gravity, such that the continuity equation  $\nabla^\mu T_{\mu\nu} = 0$  holds. Then, the energy density  $\rho_m$  and the pressure  $P_m$  obey

$$P'_m + \Psi'(\rho_m + P_m) = 0, \quad (2.9)$$

where a prime represents a derivative with respect to  $r$ .

The vector field  $A^\mu$  can be written in the form  $A^\mu = (\phi, A^i)$ , where  $\phi$  is the temporal component and  $A^i$  is the three-dimensional vector. The spatial components are decomposed as  $A_i = A_i^{(T)} + \nabla_i \chi$ , where  $A_i^{(T)}$  is the transverse component satisfying  $\nabla^i A_i^{(T)} = 0$  and  $\chi$  is the longitudinal scalar. On the spherically symmetric background with the coordinate  $(t, r, \theta, \varphi)$ , the  $\theta$  and  $\varphi$  components of  $A_i^{(T)}$  vanish, i.e.,  $A_2^{(T)} = 0$  and  $A_3^{(T)} = 0$ . The transverse condition of  $A_i^{(T)}$  gives the solution  $A_1^{(T)} = C e^\Phi / r^2$  [41]. The integration constant  $C$  is required to be 0 for the regularity at  $r = 0$ , so that  $A_1^{(T)} = 0$ . Hence the vector field can be expressed as

$$A^\mu = (\phi(r), e^{-2\Phi} \chi'(r), 0, 0). \quad (2.10)$$

On using Eq. (2.10), the term  $X$  in Eq. (2.7) is decomposed as

$$X = X_\phi + X_\chi, \quad (2.11)$$

where

$$X_\phi \equiv \frac{1}{2} e^{2\Psi} \phi'^2, \quad X_\chi \equiv -\frac{1}{2} e^{-2\Phi} \chi'^2. \quad (2.12)$$

To derive the background equations of motion, we may write the metric (2.8) in a more general form  $ds^2 = -e^{2\Psi(r)} dt^2 + e^{2\Phi(r)} dr^2 + r^2 e^{2\zeta(r)} d\Omega^2$  and express the action (2.1) in terms of  $\Psi, \Phi, \zeta, \phi, \chi$ . Varying the resulting action with respect to  $\Psi, \Phi, \zeta, \phi, \chi$  and setting  $\zeta = 0$  in the end, it follows that

$$C_1 \Psi'^2 + \left(C_2 + \frac{C_3}{r}\right) \Psi' + \left(C_4 + \frac{C_5}{r}\right) \Phi' + C_6 + \frac{C_7}{r} + \frac{C_8}{r^2} = -e^{2\Phi} \rho_m, \quad (2.13)$$

$$C_9 \Psi'^2 + \left(C_{10} + \frac{C_{11}}{r}\right) \Psi' + C_{12} + \frac{C_{13}}{r} + \frac{C_{14}}{r^2} = e^{2\Phi} P_m, \quad (2.14)$$

$$C_{15} \Psi'' + C_{16} \Phi'' + C_{17} \Psi'^2 + C_{18} \Psi' \Phi' + C_{19} \Phi'^2 + \left(C_{20} + \frac{C_{21}}{r}\right) \Psi' + \left(C_{22} + \frac{C_{23}}{r}\right) \Phi' + C_{24} + \frac{C_{25}}{r} = e^{2\Phi} P_m, \quad (2.15)$$

$$D_1 (\Psi'' + \Psi'^2) + D_2 \Psi' \Phi' + \left(D_3 + \frac{D_4}{r}\right) \Psi' + \left(D_5 + \frac{D_6}{r}\right) \Phi' + D_7 + \frac{D_8}{r} + \frac{D_9}{r^2} = 0, \quad (2.16)$$

$$D_{10} \Psi'^2 + \left(D_{11} + \frac{D_{12}}{r}\right) \Psi' + \frac{D_{13}}{r} \Phi' + D_{14} + \frac{D_{15}}{r} + \frac{D_{16}}{r^2} = 0, \quad (2.17)$$

where the coefficients  $C_{1-25}$  and  $D_{1-16}$  are given in Appendix A. Among the continuity equation (2.9) and Eqs. (2.13)-(2.17), five of them are independent. For instance, it is possible to derive Eq. (2.15) by combining other equations of motion. As a result of going beyond the domain of generalized Proca theories, the third-order spatial derivative  $\chi'''$  appears in the coefficient  $C_{24}$ .

### III. EXISTENCE OF SOLID ANGLE DEFICIT SINGULARITIES IN GLPV THEORIES

We first revisit how solid angle deficit singularities arise in GLPV theories to clarify the difference from beyond-generalized Proca theories later. The quartic GLPV theories can be recovered by taking the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$  in Eqs. (2.3)-(2.6) (i.e., without the temporal component  $\phi$ ) and by allowing the dependence on  $\chi$  as well as on  $X = -\nabla_\mu \chi \nabla^\mu \chi / 2$  in the functions  $G_{2,3,4}$  and  $f_4$ . Then, the Lagrangian densities of GLPV theories up to quartic order are given by

$$\mathcal{L}_2 = G_2(\chi, X), \quad (3.1)$$

$$\mathcal{L}_3 = G_3(\chi, X) \nabla_\mu \nabla^\mu \chi, \quad (3.2)$$

$$\mathcal{L}_4 = G_4(\chi, X) R + G_{4,X}(\chi, X) [(\nabla_\mu \nabla^\mu \chi)^2 - \nabla_\rho \nabla_\sigma \chi \nabla^\sigma \nabla^\rho \chi], \quad (3.3)$$

$$\mathcal{L}_4^N = f_4(\chi, X) \mathcal{E}_{\alpha_1 \alpha_2 \alpha_3 \gamma_4} \mathcal{E}^{\beta_1 \beta_2 \beta_3 \gamma_4} \nabla^{\alpha_1} \chi \nabla_{\beta_1} \chi \nabla^{\alpha_2} \nabla_{\beta_2} \chi \nabla^{\alpha_3} \nabla_{\beta_3} \chi. \quad (3.4)$$

To quantify the deviation from Horndeski theories, we define the parameter<sup>1</sup>

$$\alpha_H = -\frac{4f_4 X^2}{A_4}, \quad (3.5)$$

where

$$A_4 = 2XG_{4,X} - G_4 + 4f_4 X^2. \quad (3.6)$$

The quantity  $A_4$  arises after expressing the Horndeski Lagrangian in terms of ADM variables by choosing the unitary gauge [14, 15]. In general relativity we have  $G_4 = M_{\text{pl}}^2/2$  and  $f_4 = 0$  ( $M_{\text{pl}}$  is the reduced Planck mass), so that  $A_4 = -G_4 = -M_{\text{pl}}^2/2$ . One of the simplest models outside the Horndeski domain is characterized by constant functions  $A_4$  and  $-G_4$  with  $A_4 \neq -G_4$ , in which case the terms  $4f_4 X^2$  as well as  $\alpha_H$  are constants. In Ref. [21] it was shown that, for the models with non-vanishing  $\alpha_H$  at  $r = 0$ , there exists the solid angle deficit singularity with divergent  $R$ .

To see how the solid angle deficit singularity arises, we consider the model described by

$$G_2 = X, \quad G_3 = 0, \quad G_4 = \text{constant}, \quad f_4 = \frac{d_4}{X^2}, \quad (3.7)$$

where  $d_4$  is a constant. In this model the parameter  $\alpha_H$  is constant, which is related to  $d_4$ , as

$$d_4 = \frac{\alpha_H}{4(1 + \alpha_H)} G_4. \quad (3.8)$$

In GLPV theories we have  $X_\phi = 0$  and hence  $X = X_\chi$ . The non-vanishing coefficients of Eq. (2.13) are given by

$$C_5 = -4G_4 + 16X_\chi^2(5f_4 + 2X_\chi f_{4,X}) = -\frac{4G_4}{1 + \alpha_H}, \quad (3.9)$$

$$C_8 = 2(1 - e^{2\Phi})G_4 - 8X_\chi^2 f_4 = \frac{2G_4}{1 + \alpha_H} [1 - e^{2\Phi}(1 + \alpha_H)], \quad (3.10)$$

and  $C_6 = \chi'^2/2$ . Let us first derive the solution to  $\Phi$  under the assumption that  $\chi = \text{constant}$ . The justification of this assumption will be confirmed later. Then, Eq. (2.13) reduces to

$$-\frac{2G_4}{1 + \alpha_H} \left[ \frac{2\Phi'}{r} - \frac{1 - (1 + \alpha_H)e^{2\Phi}}{r^2} \right] + e^{2\Phi} \rho_m = 0. \quad (3.11)$$

Assuming that the matter density  $\rho_m$  is constant around the center of the compact body, the solution to Eq. (3.11) is given by

$$\Phi = -\frac{1}{2} \ln \left[ 1 + \alpha_H - \frac{(1 + \alpha_H)\rho_m r^2}{6G_4} \right], \quad (3.12)$$

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<sup>1</sup> The Levi-Civita tensor obeys the normalization  $\mathcal{E}^{\mu\nu\rho\sigma}\mathcal{E}_{\mu\nu\rho\sigma} = -4!$ , which is different from that of Refs. [21, 22]. The definition of  $X$  also differs, so we need to change  $X \rightarrow -2X$  and  $F_4 \rightarrow -F_4$  compared to the notations of Refs. [21, 22].

where the integration constant  $C$ , which appears as the form  $C/(2G_4r)$  in the logarithmic term of Eq. (3.12), has been set to 0 for the regularity of  $\Phi$  at  $r = 0$ . In the limit that  $r \rightarrow 0$ , there exists the non-vanishing constant  $-(1/2)\ln(1 + \alpha_H)$  in  $\Phi$  for  $\alpha_H \neq 0$ . As we will see below, this is the source for solid angle deficit singularities at  $r = 0$ .

The solution (3.12) was obtained by assuming that  $\chi$  is constant. In the following we iteratively derive the solutions to  $\Phi, \Psi, \chi$  by expanding them around the center of the spherically symmetric body, as

$$\Phi(r) = \Phi_0 + \sum_{i=1}^{\infty} \Phi_i r^i, \quad \Psi(r) = \Psi_0 + \sum_{i=1}^{\infty} \Psi_i r^i, \quad \chi(r) = \chi_0 + \sum_{i=2}^{\infty} \chi_i r^i, \quad (3.13)$$

where  $\Phi_0, \Phi_i, \Psi_0, \Psi_i, \chi_0, \chi_i$  are constants. The field  $\chi(r)$  should satisfy the regular boundary condition  $\chi'(0) = 0$ , so the term  $\chi_1 r$  is absent. The matter density  $\rho_m(r)$  can be also expanded around  $r = 0$ , but the variation of  $\rho_m(r)$  does not affect the discussion of solid angle deficit singularities [21, 22]. Hence it is sufficient to consider the case of constant  $\rho_m$ . In this case, Eq. (2.9) is integrated to give

$$P_m(r) = -\rho_m + \rho_s e^{-\Psi(r)}, \quad (3.14)$$

where  $\rho_s$  is a constant.

The equations of motion in shift-symmetric GLPV theories can be derived by setting  $\phi = 0$  in Eqs. (2.13)-(2.17). Since Eq. (2.16) is decoupled from the system, three of Eqs. (2.13)-(2.17) are independent. Substituting Eqs. (3.13)-(3.14) into Eqs. (2.13), (2.14), and (2.17) for the model (3.7), we can iteratively obtain the following solutions

$$\Phi(r) = -\frac{1}{2} \ln(1 + \alpha_H) + \frac{\rho_m}{12G_4} r^2 + \mathcal{O}(r^4), \quad (3.15)$$

$$\Psi(r) = \Psi_0 - \frac{2\rho_m e^{\Psi_0} - 3\rho_s}{24G_4 e^{\Psi_0}} r^2 + \mathcal{O}(r^4), \quad (3.16)$$

$$\chi(r) = \chi_0. \quad (3.17)$$

Thus, the assumption that  $\chi(r) = \text{constant}$  used for the derivation of the solution (3.12) is justified. In fact, expansion of the solution (3.12) around  $r = 0$  leads to Eq. (3.15). On using Eqs. (3.15) and (3.16), the Ricci scalar reads

$$R = -\frac{2\alpha_H}{r^2} + \frac{(4\rho_m e^{\Psi_0} - 3\rho_s)(1 + \alpha_H)}{2G_4 e^{\Psi_0}} + \mathcal{O}(r^2). \quad (3.18)$$

If the parameter  $\alpha_H$  does not vanish at  $r = 0$ , there is the solid angle deficit singularity with divergent  $R$ .

The model (3.7) is the simplest one in which the solid angle deficit singularity is present. In Refs. [21, 22], the authors considered more general cases in which additional functions  $G_2, G_3, G_4, f_4$  to the model (3.7) are taken into account. Provided  $\alpha_H \neq 0$  at  $r = 0$ , it was shown that these additional contributions do not modify the existence of solid angle deficit singularities. To avoid the appearance of solid angle deficit singularities, we need to construct models in which  $\alpha_H$  vanishes at  $r = 0$ . For example, the model with the functions  $f_4 = \text{constant}$  and  $G_4 = M_{\text{pl}}^2 F(\chi)/2 + b_4 X^2$  ( $F(\chi)$  is a function of  $\chi$  and  $b_4$  is a constant) gives  $\alpha_H = 0$  at  $r = 0$  due to the boundary condition  $\chi'(0) = 0$ , while  $\alpha_H$  does not vanish for  $r > 0$ . In such models, not only the problem of solid angle deficit singularities is absent, but also the Vainshtein screening can be at work outside the compact body [21, 22].

#### IV. ABSENCE OF SOLID ANGLE DEFICIT SINGULARITIES IN BEYOND-GENERALIZED PROCA THEORIES

The appearance of solid angle deficit singularities in GLPV theories is intrinsically related to the non-vanishing term  $-(1/2)\ln(1 + \alpha_H)$  in Eq. (3.15). In fact, the regularities of  $R$  and the curvature scalars  $R_{\mu\nu}R^{\mu\nu}$ ,  $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$  generally require the three conditions  $\Phi_0 = 0$ ,  $\Phi_1 = 0$ , and  $\Psi_1 = 0$  in Eq. (3.13) [22]. In other words, if the gravitational potentials are expanded around  $r = 0$  as

$$\Phi(r) = \Phi_2 r^2 + \mathcal{O}(r^4), \quad (4.1)$$

$$\Psi(r) = \Psi_0 + \Psi_2 r^2 + \mathcal{O}(r^4), \quad (4.2)$$

the solid angle deficit singularity is absent.

In what follows, we show that solid angle deficit singularities do not arise in beyond-generalized Proca theories (2.1) by virtue of the existence of the temporal vector component  $\phi$  besides the longitudinal component  $\chi$ . To characterize the deviation from second-order generalized Proca theories, we define the quantity analogous to Eq. (3.5), i.e.,

$$\alpha_P = \frac{4f_4 X^2}{G_4 - 2XG_{4,X} - 4f_4 X^2}. \quad (4.3)$$

The regular boundary conditions of the vector field at  $r = 0$  are given by

$$\phi'(0) = 0, \quad \chi'(0) = 0. \quad (4.4)$$

Since  $\phi$  and  $\chi$  stay nearly constants around  $r = 0$ , we have that

$$X_\chi(r \rightarrow 0) = 0, \quad X_\phi(r \rightarrow 0) = \frac{1}{2}e^{2\Psi}\phi^2 = \text{constant}. \quad (4.5)$$

The key difference from GLPV theories is that, provided  $\phi \neq 0$ ,  $X$  approaches the non-vanishing constant  $X_\phi$  as  $r \rightarrow 0$ .

To understand the difference from GLPV theories, we begin with the following model

$$G_2 = b_2 X, \quad G_3 = 0, \quad G_4 = \text{constant}, \quad g_4 = \text{constant}, \quad f_4 = \frac{\alpha_P}{4(1 + \alpha_P)} \frac{G_4}{X^2}, \quad (4.6)$$

where  $b_2$  and  $\alpha_P$  are constants. This is analogous to the model (3.7) in which solid angle deficit singularities arise in GLPV theories. Then, the coefficients  $C_5$  and  $C_8$  in Eq. (2.13) are given, respectively, by

$$C_5 = -4G_4 + \frac{4\alpha_P G_4}{1 + \alpha_P} \frac{X_\chi(3X_\phi + X_\chi)}{(X_\phi + X_\chi)^2}, \quad (4.7)$$

$$C_8 = 2(1 - e^{2\Phi})G_4 - \frac{2\alpha_P G_4}{1 + \alpha_P} \frac{X_\chi}{X_\phi + X_\chi}. \quad (4.8)$$

On account of the relation (4.5), the last two terms of Eqs. (4.7) and (4.8) vanish for  $r \rightarrow 0$ , so that  $C_5 \rightarrow -4G_4$  and  $C_8 \rightarrow 2(1 - e^{2\Phi})G_4$ . Unlike GLPV theories, the coefficients  $C_5$  and  $C_8$  do not contain the  $\alpha_P$  term. In GLPV theories the term  $\alpha_H$  arising in the square bracket of  $C_8$  in Eq. (3.10) is the main source for the existence of solid angle deficit singularities. In beyond-generalized Proca theories, the absence of the terms  $\alpha_P$  in  $C_5$  and  $C_8$  implies that solid angle deficit singularities may not arise at  $r = 0$ .

To see explicitly the absence of solid angle deficit singularities for the model (4.6), we expand the temporal vector component around  $r = 0$ , as

$$\phi(r) = \phi_0 + \sum_{i=2}^{\infty} \phi_i r^i, \quad (4.9)$$

where  $\phi_0, \phi_i$  are constants. The gravitational potentials  $\Phi, \Psi$  and the longitudinal scalar  $\chi$  are expanded in the same way as Eq. (3.13). For constant density  $\rho_m$  around  $r = 0$ , the matter pressure  $P_m$  is given by Eq. (3.14). Unlike GLPV theories in which  $\phi$  exactly vanishes, Eq. (2.16) plays an important role to determine the field profile of  $\phi$ . Substituting Eqs. (3.13) and (4.9) with Eq. (3.14) into Eqs. (2.13), (2.14), (2.16), (2.17) and solving them iteratively, we obtain the following solutions

$$\Phi(r) = \frac{2\rho_m + b_2 e^{2\Psi_0} \phi_0^2}{24G_4} r^2 + \mathcal{O}(r^4), \quad (4.10)$$

$$\Psi(r) = \Psi_0 - \frac{2\rho_m - 3e^{-\Psi_0} \rho_s - 2b_2 e^{2\Psi_0} \phi_0^2}{24G_4} r^2 + \mathcal{O}(r^4), \quad (4.11)$$

$$\phi(r) = \phi_0 + \frac{\phi_0}{6} \left( \frac{b_2}{1 - 2g_4} + \frac{2\rho_m - 3e^{-\Psi_0} \rho_s - 2b_2 e^{2\Psi_0} \phi_0^2}{2G_4} \right) r^2 + \mathcal{O}(r^4), \quad (4.12)$$

$$\chi(r) = \chi_0, \quad (4.13)$$

which are expanded up to second order in  $r$ . Compared to Eq. (3.15), there is no constant term containing  $\alpha_P$  on the r.h.s. of Eq. (4.10). The solutions (4.10) and (4.11) are of the same forms as Eqs. (4.1) and (4.2) respectively, so we do not have solid angle deficit singularities at  $r = 0$ . In fact, the Ricci scalar is given by

$$R = \frac{4\rho_m - 3e^{-\Psi_0} \rho_c - b_2 e^{2\Psi_0} \phi_0^2}{2G_4} + \mathcal{O}(r), \quad (4.14)$$

which is finite at  $r = 0$ .

So far we have shown the absence of solid angle deficit singularities for the model described by Eq. (4.6), but this property generally holds even without restricting the functional forms of  $G_{2,3,4}, g_4, f_4$ . To prove this, we use the



relations (4.5) with the boundary conditions (4.4). Then, around  $r = 0$ , Eqs. (2.13) and (2.14) reduce, respectively, to

$$-2(G_4 - 2X_\phi G_{4,X}) \left( \frac{2\Phi'}{r} - \frac{1 - e^{2\Phi}}{r^2} \right) + C_1 \Psi'^2 + C_6 + e^{2\Phi} \rho_m \simeq 0, \quad (4.15)$$

$$4(G_4 + 2X_\phi G_{4,X}) \frac{\Psi'}{r} + 2(1 - e^{2\Phi}) \frac{G_4}{r^2} + C_9 \Psi'^2 + C_{12} + e^{2\Phi} (\rho_m - \rho_s e^{-\Psi}) \simeq 0, \quad (4.16)$$

where  $C_1, C_6, C_9, C_{12}$  are constants. Substituting the expanded gravitational potentials (3.13) into Eq. (4.15), it follows that the two terms  $2(1 - e^{2\Phi_0})(G_4 - 2X_\phi G_{4,X})/r^2$  and  $-4\Phi_1(1 + e^{2\Phi_0})(G_4 - 2X_\phi G_{4,X})/r$  are required to vanish. As long as  $G_4 \neq 2X_\phi G_{4,X}$ , we have

$$\Phi_0 = 0, \quad \Phi_1 = 0. \quad (4.17)$$

From Eq. (4.16) we also find that there is one term  $4(G_4 + 2X_\phi G_{4,X})\Psi_1/r$  that must vanish. Provided  $G_4 \neq -2X_\phi G_{4,X}$ , we obtain

$$\Psi_1 = 0. \quad (4.18)$$

Then, the gravitational potentials  $\Phi(r)$  and  $\Psi(r)$  reduce to the forms (4.1) and (4.2) around  $r = 0$ , respectively, so the solid angle deficit singularity is absent. The specific case satisfying  $G_4 = 2X_\phi G_{4,X} = -2X_\phi G_{4,X}$  corresponds to  $G_4 = 0$ , so this does not correspond to a realistic situation where the general relativistic behavior is recovered. Hence the absence of solid angle deficit singularities is very generic in quartic-order beyond-generalized Proca theories.

## V. VAINSHTEIN MECHANISM

In this section we discuss how the new term  $\mathcal{L}_4^N$  in beyond-generalized Proca theories affects the screening mechanism in second-order generalized Proca theories. For concreteness, we study the theories described by

$$G_2(X) = m^2 X, \quad G_3(X) = \beta_3 X, \quad G_4(X) = \frac{M_{\text{pl}}^2}{2} + \beta_4 X^2, \quad g_4(X) = 0, \quad f_4(X) = d_4 X^n, \quad (5.1)$$

where  $m^2, \beta_3, \beta_4, d_4, n$  are constants. We assume that  $|n|$  is of the order of unity. When  $d_4 = 0$ , the theories reduce to vector Galileons [36]. In principle we can consider more general functions of  $G_{2,3,4}, g_4$ , but the above theories are sufficient to address the problem of how the new interaction  $f_4(X)$  affects the Vainshtein mechanism mediated by vector Galleons (see Refs. [47] for the Vainshtein mechanism mediated by scalar Galileons).

For the functions (5.1) the vector-field equations of motion (2.16) and (2.17) are given, respectively, by

$$\begin{aligned} & \frac{1}{r^2} \frac{d}{dr} (r^2 \phi') - e^{2\Phi} m^2 \phi + 2\phi (\Psi'' + \Psi'^2 - \Psi' \Phi') + \left( 3\phi' + \frac{4\phi}{r} \right) \Psi' - \phi' \Phi' - \beta_3 \phi \left[ \frac{1}{r^2} \frac{d}{dr} (r^2 \chi') + (\Psi' - \Phi') \chi' \right] \\ & - \frac{2\beta_4 e^{-2\Phi} \phi}{r^2} [4r\chi'\chi'' + e^{2\Psi+2\Phi} \phi^2 (e^{2\Phi} - 1 + 2r\Phi') - \chi'^2 \{e^{2\Phi} - 3 + 2r(3\Phi' - 2\Psi')\}] \\ & - \frac{2^{2-n} d_4 e^{-2\Phi} \phi \chi'}{r^2} (e^{2\Psi} \phi^2 - e^{-2\Phi} \chi'^2)^n [2(2+n)r\chi'' + \{2+n-r\Phi'(5+2n)+r\Psi'(3+2n)\}\chi'] = 0, \quad (5.2) \\ & m^2 \chi' + \beta_3 \left[ e^{2\Psi} (\phi\phi' + \phi^2 \Psi') + e^{-2\Phi} \chi'^2 \left( \frac{2}{r} + \Psi' \right) \right] \\ & + \frac{2\beta_4 \chi'}{r} \left[ e^{2\Psi} \frac{\phi^2}{r} (1 - e^{-2\Phi}) + 2e^{2\Psi-2\Phi} (2\phi\phi' + \phi^2 \Psi') - e^{-2\Phi} \frac{\chi'^2}{r} (1 - 3e^{-2\Phi} - 6r\Psi'e^{-2\Phi}) \right] \\ & + \frac{2^{2-n} d_4 e^{-4\Phi} \chi'}{r^2} (e^{2\Psi} \phi^2 - e^{-2\Phi} \chi'^2)^n [e^{2\Psi+2\Phi} \phi \{2(2+n)\phi' + (2n\Psi' + 3\Psi' - \Phi')\phi\}r + (2+n)(1+2r\Psi')\chi'^2] \\ & = 0. \quad (5.3) \end{aligned}$$

The vector mass squared  $m^2$  can be either positive or negative. Applying the above model to dark energy, it is natural to consider the mass scale  $|m|$  of the order of  $H_0 \approx 10^{-33}$  eV [39]. For the study of spherically symmetric solutions in the solar system, we take the limit  $m^2 \rightarrow 0$  in the following discussion.

We consider a compact body with the radius  $r_*$  and the constant density  $\rho_0$ , i.e.,  $\rho_m(r) = \rho_0$  for  $r < r_*$  and  $\rho_m(r) = 0$  for  $r > r_*$ . We also employ the approximation of weak gravity under which the Schwarzschild radius  $r_g \approx \rho_0 r_*^3 / M_{\text{pl}}^2$  of the body is much smaller than  $r_*$ , i.e.,

$$\Phi_* \equiv \frac{\rho_0 r_*^2}{M_{\text{pl}}^2} \ll 1. \quad (5.4)$$

The general relativistic solutions to  $\Phi$  and  $\Psi$  in the absence of the vector field can be derived by setting  $G_2 = G_3 = 0$ ,  $G_4 = M_{\text{pl}}^2/2$ , and  $\phi = \chi' = 0$  in Eqs. (2.13) and (2.14). They are given, respectively, by [41]

$$\Phi_{\text{GR}} = \frac{\Phi_*}{6} \frac{r^2}{r_*^2}, \quad \Psi_{\text{GR}} = \frac{\Phi_*}{12} \left( \frac{r^2}{r_*^2} - 3 \right), \quad (5.5)$$

for  $r < r_*$ , and

$$\Phi_{\text{GR}} = \frac{\Phi_* r_*}{6r}, \quad \Psi_{\text{GR}} = -\frac{\Phi_* r_*}{6r}, \quad (5.6)$$

for  $r > r_*$ . In the presence of the vector field coupled to gravity, these solutions are subject to modifications. As long as the Vainshtein mechanism is at work, the corrections to  $\Phi_{\text{GR}}$  and  $\Psi_{\text{GR}}$  should be small. For the theories given by the functions (5.1), we shall estimate the corrections to Eqs. (5.5) and (5.6).

### A. $\beta_3 = 0$

If  $\beta_3 = 0$ , then Eq. (5.3) admits the solution

$$\chi' = 0, \quad (5.7)$$

for which the longitudinal vector component  $\chi$  stays constant. In this case the last line on the l.h.s. of Eq. (5.2) vanishes identically, so the Lagrangian density  $\mathcal{L}_4^{\text{N}}$  does not give rise to any contribution to the profile of  $\phi$ . Hence the solution of  $\phi$  is similar to the one already derived in Ref. [41] for  $\chi' = 0$ .

Following Ref. [41], we search for solutions where  $\phi$  stays nearly constant around a constant  $\phi_0$ , such that

$$\phi(r) = \phi_0 + f(r), \quad |f(r)| \ll |\phi_0|, \quad (5.8)$$

where  $f(r)$  is a function of  $r$ . For the integration of Eq. (5.2), the temporal component  $\phi$  is approximated as  $\phi_0$ . Under the approximation of weak gravity, we neglect the terms like  $\phi'\Phi'$  relative to the first contribution on the l.h.s. of Eq. (5.2). We also substitute the gravitational potentials (5.5) and (5.6) into Eq. (5.2) to derive the leading-order solution to  $\phi(r)$ . In doing so, second-order gravitational potentials like  $2\phi\Psi'^2$  are neglected relative to their first-order contribution.

Employing this prescription for  $r < r_*$ , Eq. (5.2) reduces to

$$\frac{d}{dr} (r^2 \phi') + \phi_0 \Phi_* (1 - 2\beta_4 \phi_0^2) \frac{r^2}{r_*^2} \simeq 0. \quad (5.9)$$

Under the boundary condition  $\phi'(0) = 0$ , the integrated solution to Eq. (5.9) is given by

$$\phi'(r) \simeq -\frac{\phi_0 \Phi_* (1 - 2\beta_4 \phi_0^2)}{3r_*^2} r. \quad (5.10)$$

We then obtain  $\phi(r)$  in the form (5.8) with  $f(r) = -\phi_0 \Phi_* (1 - 2\beta_4 \phi_0^2) r^2 / (6r_*^2)$ . Provided that the term  $1 - 2\beta_4 \phi_0^2$  is at most of the order of 1, the condition  $|f(r)| \ll |\phi_0|$  is well satisfied.

For  $r > r_*$ , substituting Eq. (5.6) into Eq. (5.2) and picking up first-order contributions of  $\Phi$  and  $\Psi$ , it follows that

$$\frac{d}{dr} (r^2 \phi') \simeq 0. \quad (5.11)$$

The integrated solution to Eq. (5.11) is given by  $\phi'(r) = C/r^2$ , where the constant  $C$  is known by matching solutions at  $r = r_*$ . Then, the resulting solution for the radius  $r > r_*$  reads

$$\phi'(r) \simeq -\frac{\phi_0 \Phi_* (1 - 2\beta_4 \phi_0^2)}{3r^2} r_*. \quad (5.12)$$



Unlike Ref. [41], we have taken into account the term  $2\beta_4\phi_0^2$  without necessarily assuming the condition  $|\beta_4|\phi_0^2 \ll 1$ .

To derive corrections to the gravitational potentials (5.6) induced by the temporal vector component  $\phi$  outside the compact body, we substitute Eqs. (5.7) and (5.12) into Eqs. (2.13) and (2.14) under the approximation of weak gravity. Then, for  $r > r_*$ , the gravitational potentials approximately obey

$$\frac{2M_{\text{pl}}^2}{r}\Phi' + \frac{2M_{\text{pl}}^2}{r^2}\Phi \simeq -\frac{\phi_0^2\Phi_*r_*^2}{18r^4}(1 - 4\beta_4^2\phi_0^4), \quad (5.13)$$

$$\frac{2M_{\text{pl}}^2}{r}\Psi' - \frac{2M_{\text{pl}}^2}{r^2}\Phi \simeq \frac{\phi_0^2\Phi_*r_*}{18r^4} [12\beta_4\phi_0^2r(1 - 4\beta_4\phi_0^2) + \Phi_*r_*], \quad (5.14)$$

which are integrated to give

$$\Phi(r) \simeq \frac{\Phi_*r_*}{6r} \left[ 1 + \frac{\phi_0^2\Phi_*r_*}{6M_{\text{pl}}^2r}(1 - 4\beta_4^2\phi_0^4) \right], \quad (5.15)$$

$$\Psi(r) \simeq -\frac{\Phi_*r_*}{6r} \left[ 1 + \frac{2\beta_4\phi_0^4}{M_{\text{pl}}^2}(1 - 4\beta_4\phi_0^2) + \frac{\phi_0^2\Phi_*r_*}{6M_{\text{pl}}^2r} \right]. \quad (5.16)$$

As long as the correction terms in the square brackets of Eqs. (5.15) and (5.16) are much smaller than 1, the post-Newtonian parameter  $\gamma = -\Phi/\Psi$  reduces to

$$\gamma \simeq 1 - \frac{2\beta_4\phi_0^4}{M_{\text{pl}}^2}(1 - 4\beta_4\phi_0^2). \quad (5.17)$$

On using the local gravity constraint  $|\gamma - 1| < 2.3 \times 10^{-5}$  [48], we obtain the bound

$$|\beta_4\phi_0^4(1 - 4\beta_4\phi_0^2)| < 1 \times 10^{-5}M_{\text{pl}}^2. \quad (5.18)$$

If  $|\beta_4|\phi_0^2 \ll 1$ , then the condition (5.18) translates to  $|\beta_4|\phi_0^4 < 1 \times 10^{-5}M_{\text{pl}}^2$ . For  $\phi_0$  of the order of  $M_{\text{pl}}$  the latter condition gives the bound  $|\beta_4| \lesssim 10^{-5}/M_{\text{pl}}^2$ , in which case the condition  $|\beta_4|\phi_0^2 \ll 1$  is automatically satisfied. For the theories with  $\beta_3 = 0$  the perfect screening of the longitudinal mode occurs even with the new term  $\mathcal{L}_4^{\text{N}}$ , so only the temporal vector component  $\phi$  gives rise to corrections to gravitational potentials.

## B. $\beta_3 \neq 0$

In the presence of the cubic interaction  $G_3(X)$ ,  $\chi'$  does not generally vanish. If the effect of the coupling  $\beta_3$  always dominates over those of  $\beta_4$  and  $d_4$ , then the resulting field profiles of  $\phi$  and  $\chi$  are practically the same as those derived in Ref. [41] for  $\beta_4 = 0 = d_4$ . Since we are interested in how the new interaction  $f_4(X) = d_4X^n$  affects the screening mechanism, we focus on the case in which effects of the couplings  $d_4$  and  $\beta_4$  dominate over that of  $\beta_3$  at least for the distance relevant to the solar system.

We employ the weak-gravity approximation and assume the condition

$$\chi'^2 \ll \phi^2, \quad (5.19)$$

which can be justified after deriving the solution to  $\chi'$ . Then, Eqs. (5.2) and (5.3) reduce, respectively, to

$$\begin{aligned} & \frac{d}{dr}(r^2\phi') - \beta_3\phi\frac{d}{dr}(r^2\chi') - 4\beta_4\phi\frac{d}{dr}(r\chi'^2) + 2\phi\frac{d}{dr}(r^2\Psi') - 4\beta_4\phi^3(\Phi + r\Phi') \\ & - 2^{2-n}d_4\phi^{2n+1}\chi'[(2+n)(2r\chi'' + \chi') + r\chi'\{(3+2n)\Psi' - (5+2n)\Phi'\}] \simeq 0, \end{aligned} \quad (5.20)$$

$$\chi' \simeq -\frac{\beta_3r(r\phi\phi' + 2\chi'^2 + r\phi^2\Psi')}{4\beta_4[2r\phi\phi' + \chi'^2 + \phi^2(\Phi + r\Phi')] + 2^{2-n}d_4\phi^{2n}[(2+n)(2r\phi\phi' + \chi'^2) + r\phi^2(2n\Psi' + 3\Psi' - \Phi)]}. \quad (5.21)$$

In what follows, we derive the solutions to Eqs. (5.20) and (5.21) for several different radii. As before, we search for solutions of the temporal component in the form (5.8).

1.  $r < r_*$

For the distance  $r < r_*$ , the leading-order solution to  $\phi'(r)$  can be derived by taking the limit  $\chi' \rightarrow 0$  in Eq. (5.20). This is equivalent to Eq. (5.10), i.e.,

$$\phi'(r) \simeq -\frac{\rho_0 \phi_0 (1 - 2\beta_4 \phi_0^2)}{3M_{\text{pl}}^2} r. \quad (5.22)$$

Provided that the condition

$$\chi'^2 \ll r|\phi\phi'| \quad (5.23)$$

is satisfied, substitutions of Eqs. (5.5) and (5.22) into Eq. (5.21) lead to

$$\chi'(r) \simeq -\frac{\beta_3(1 - 4\beta_4 \phi_0^2)}{8\beta_4(1 - 4\beta_4 \phi_0^2) + 2^{2-n} d_4 \phi_0^{2n} [7 + 2n - 8(2+n)\beta_4 \phi_0^2]} r. \quad (5.24)$$

For  $|\beta_4| \phi_0^2 \ll 1$ , the condition (5.23) translates to

$$\varepsilon \equiv \frac{3\beta_3^2 M_{\text{pl}}^2}{[8\beta_4 + 2^{2-n} d_4 (7 + 2n) \phi_0^{2n}]^2 \rho_0 \phi_0^2} \ll 1. \quad (5.25)$$

Since the solution (5.22) satisfies the inequality  $|r\phi'| \ll |\phi|$ , the assumption (5.19) is justified under the condition (5.23).

Substituting Eq. (5.24) into Eq. (5.20) with  $|\beta_4| \phi_0^2 \ll 1$ , the next-to-leading order solution to  $\phi'(r)$  is given by

$$\phi'(r) \simeq -\frac{\rho_0 \phi_0}{3M_{\text{pl}}^2} (1 + \delta_1) r, \quad \delta_1 \equiv \frac{3 \cdot 2^{n-2} \beta_3^2 M_{\text{pl}}^2 [2^n \beta_4 + (5+n) d_4 \phi_0^{2n}]}{[2^{n+1} \beta_4 + (7+2n) d_4 \phi_0^{2n}]^2 \rho_0}. \quad (5.26)$$

Provided that

$$2^{-n} |d_4| \phi_0^{2n+2} \lesssim 1, \quad (5.27)$$

$|\delta_1|$  is at most of the order of  $\varepsilon$ . This means that, under the condition (5.25), the correction  $\delta_1$  to the leading-order solution (5.22) can be neglected. The above results show that, for  $r < r_*$ , both  $|\chi'(r)|$  and  $|\phi'(r)|$  linearly grow in  $r$ .

2.  $r_* < r < r_t$

For  $r > r_*$  the leading-order gravitational potentials are given by Eq. (5.6), so this causes the decrease of  $|\phi'(r)|$  as in Eq. (5.12). Since there is a transition radius  $r_t$  at which  $\chi'^2$  grows to the same order as  $r|\phi\phi'|$ , we first study the behavior of solutions for the distance  $r_* < r < r_t$ . In this regime, we can neglect the terms  $\chi'^2$  appearing on the r.h.s. of Eq. (5.21). Plugging Eq. (5.6) into Eq. (5.21), it follows that

$$\chi'(r) \simeq -\frac{\beta_3}{8[\beta_4 + 2^{-n}(2+n)d_4 \phi_0^{2n}]} r. \quad (5.28)$$

Substituting this into Eq. (5.20), we obtain the integrated solution

$$r^2 \phi'(r) + \frac{2^{n-4} \beta_3^2 \phi_0 r^3}{2^n \beta_4 + (2+n) d_4 \phi_0^{2n}} = \mathcal{C}. \quad (5.29)$$

The constant  $\mathcal{C}$  is fixed by matching Eq. (5.29) with Eq. (5.22) at  $r = r_*$ . The resulting solution to  $\phi'(r)$  for the distance  $r_* < r < r_t$  is given by

$$\phi'(r) \simeq -\frac{\rho_0 \phi_0 r_*^3}{3M_{\text{pl}}^2 r^2} \left[ 1 - 2\beta_4 \phi_0^2 + \delta_2 \left( \frac{r^3}{r_*^3} - 1 \right) \right], \quad \delta_2 \equiv \frac{3 \cdot 2^{n-4} \beta_3^2 M_{\text{pl}}^2}{[2^n \beta_4 + (2+n) d_4 \phi_0^{2n}] \rho_0}. \quad (5.30)$$

The quantity  $|\delta_2|$  is as small as  $|\delta_1|$  ( $\lesssim \varepsilon \ll 1$ ). Provided that  $|\delta_2| r^3 / r_*^3 \ll 1$ ,  $|\phi'(r)|$  decreases in proportion to  $r^{-2}$ .

The growth of  $|\chi'(r)|$  saturates for the distance  $r_t$  at which the numerator of Eq. (5.21) is close to 0. Employing the solutions (5.28) and (5.30) and neglecting the term  $\delta_2(r^3/r_*^3 - 1)$ , the transition radius can be estimated as

$$r_t = r_* \left\{ \frac{16(1 - 4\beta_4 \phi_0^2)}{\varepsilon} \left[ \frac{\beta_4 + 2^{-n}(2+n)d_4 \phi_0^{2n}}{8\beta_4 + 2^{2-n}(7+2n)d_4 \phi_0^{2n}} \right]^2 \right\}^{1/3}. \quad (5.31)$$

For  $|\beta_4| \phi_0^2 \ll 1$  we have  $r_t \approx r_*/\varepsilon^{1/3}$ , so  $r_t$  is larger than  $r_*$  for  $\varepsilon \ll 1$ .

### 3. $r > r_t$

For the distance  $r > r_t$ , the growth of  $|\chi'(r)|$  saturates in such a way that the numerator on the r.h.s. of Eq. (5.21) is close to 0, i.e.,

$$r\phi\phi' + 2\chi'^2 + \frac{\rho_0\phi^2 r_*^3}{6M_{\text{pl}}^2 r} \simeq 0. \quad (5.32)$$

In this case, the terms associated with the coupling  $\beta_3$  dominate over those related to  $\beta_4$  and  $d_4$  in Eq. (5.21), e.g.,  $|\beta_3 r| \gg |\beta_4 \chi'|$  and  $|\beta_3 r| \gg |d_4 \phi^{2n} \chi'|$ . Then, the terms containing  $\beta_4$  and  $d_4$  in Eq. (5.20) can be neglected relative to the  $\beta_3$ -dependent term. This leads to the integrated solution

$$r^2\phi' - \beta_3\phi_0 r^2\chi' \simeq -\frac{\rho_0\phi_0 r_*^3}{3M_{\text{pl}}^2}(1 - 2\beta_4\phi_0^2). \quad (5.33)$$

The r.h.s. of Eq. (5.33) corresponds to the integration constant determined by matching the solutions at  $r = r_t$ . We can explicitly solve Eqs. (5.32) and (5.33) for  $\phi'(r)$  and  $\chi'(r)$ , respectively, as

$$\phi'(r) \simeq -\frac{\rho_0\phi_0 r_*^3}{3M_{\text{pl}}^2 r^2} \mathcal{G}(\eta), \quad (5.34)$$

$$\chi'(r) \simeq \pm \sqrt{\frac{\rho_0\phi_0^2 r_*^3}{6M_{\text{pl}}^2 r} \left[ \mathcal{G}(\eta) - \frac{1}{2} \right]}, \quad (5.35)$$

where

$$\eta \equiv \frac{3\beta_3^2\phi_0^2 M_{\text{pl}}^2}{4\rho_0} \frac{r^3}{r_*^3} = 4\phi_0^4(1 - 4\beta_4\phi_0^2) [\beta_4 + 2^{-n}(2+n)d_4\phi_0^{2n}]^2 \frac{r^3}{r_t^3}, \quad (5.36)$$

$$\mathcal{G}(\eta) \equiv (1 + \eta - 2\beta_4\phi_0^2) \left[ 1 - \sqrt{1 - \frac{(1 - 2\beta_4\phi_0^2)^2 + \eta}{(1 + \eta - 2\beta_4\phi_0^2)^2}} \right]. \quad (5.37)$$

Provided that

$$\delta \equiv 4\phi_0^4(1 - 4\beta_4\phi_0^2) [\beta_4 + 2^{-n}(2+n)d_4\phi_0^{2n}]^2 \ll 1, \quad (5.38)$$

the behavior of solutions changes at the distance  $r_v$  satisfying  $\eta = 1$ , i.e.,

$$r_v = \frac{r_t}{\delta^{1/3}}. \quad (5.39)$$

From the first equality of Eq. (5.36) the distance  $r_v$  itself does not depend on  $\beta_4$  and  $d_4$ , but the ratio  $r_v/r_t$  is dependent on these couplings.

For  $r \ll r_v$  the field profiles read

$$\phi'(r) \simeq -\frac{\rho_0\phi_0 r_*^3}{3M_{\text{pl}}^2 r^2}(1 - 2\beta_4\phi_0^2), \quad \chi'(r) \simeq \pm \sqrt{\frac{\rho_0\phi_0^2 r_*^3}{12M_{\text{pl}}^2 r}(1 - 4\beta_4\phi_0^2)}, \quad (5.40)$$

whereas, for  $r \gg r_v$ , we have

$$\phi'(r) \simeq -\frac{\rho_0\phi_0 r_*^3}{6M_{\text{pl}}^2 r^2}, \quad \chi'(r) \simeq \pm \frac{\rho_0 r_*^3}{6\beta_3 M_{\text{pl}}^2 r^2}, \quad (5.41)$$

both of which do not contain the coupling  $d_4$ .

### 4. Gravitational potentials for $r > r_*$

The structure of the field profiles derived above is similar to that obtained in Ref. [41] for  $d_4 = 0$ , apart from the difference of some coefficients of  $\chi'(r)$ . In the limit that  $\beta_3 \rightarrow 0$ , the values of  $\chi'(r)$  given in Eqs. (5.24) and (5.28)

vanish with  $r_t \approx r_*/\varepsilon^{1/3} \rightarrow \infty$ . Hence, for small  $\beta_3$ , the effect of the new coupling  $d_4$  on the longitudinal scalar is unimportant.

The equations of motion (2.13) and (2.14) of gravitational potentials contain  $f_4$ -dependent terms. For  $r > r_*$  we shall estimate such contributions to the leading-order gravitational potentials (5.6). We also assume that the term  $|\beta_4|\phi_0^2$  is much smaller than 1.

In the regime  $r_* < r < r_t$ , we substitute the solutions (5.28) and (5.30) into Eqs. (2.13) and (2.14) by neglecting the  $\delta_2(r^3/r_*^3 - 1)$  term and integrate them under the condition (5.19). This leads to the following solutions

$$\Phi(r) \simeq \frac{\Phi_* r_*}{6r} \left[ 1 + \frac{\phi_0^2 \Phi_*}{6M_{\text{pl}}^2 x} - \frac{2^{n+3} \beta_4 \phi_0^4 \epsilon_0 x^3}{M_{\text{pl}}^2} - \frac{8d_4(2n+3)\phi_0^{2n+4} \epsilon_0 x^3}{M_{\text{pl}}^2} \right], \quad (5.42)$$

$$\Psi(r) \simeq -\frac{\Phi_* r_*}{6r} \left[ 1 + \frac{\phi_0^2 \Phi_*}{6M_{\text{pl}}^2 x} + \frac{2\beta_4 \phi_0^4 (1 + 2^{n+2} \epsilon_0 x^3)}{M_{\text{pl}}^2} + \frac{8d_4(n+3)\phi_0^{2n+4} \epsilon_0 x^3}{M_{\text{pl}}^2} \right], \quad (5.43)$$

where  $x \equiv r/r_*$  and

$$\epsilon_0 \equiv \frac{3\beta_3^2 M_{\text{pl}}^2}{2^{n+8} [\beta_4 + 2^{-n} d_4 (2+n) \phi_0^{2n}]^2 \rho_0 \phi_0^2}. \quad (5.44)$$

In deriving the above solutions, we have neglected the term  $\Phi_* \epsilon_0 x^2$  relative to 1. The post-Newtonian parameter  $\gamma = -\Phi/\Psi$  reduces to

$$\gamma \simeq 1 - \frac{2\beta_4 \phi_0^4 (2^{n+3} \epsilon_0 x^3 + 1)}{M_{\text{pl}}^2} - \frac{24d_4(n+2)\phi_0^{2n+4} \epsilon_0 x^3}{M_{\text{pl}}^2}. \quad (5.45)$$

For  $n = -2$ , which corresponds to the case in which solid angle deficit singularities appear in the scalar limit  $A^\mu \rightarrow \nabla^\mu \chi$  (i.e., GLPV theories), the effect of the coupling  $d_4$  appears in  $\Phi(r)$  and  $\Psi(r)$ , but it disappears in  $\gamma$ . Since the term  $2\epsilon_0 x^3$  is at most of the order of 1 at  $r = r_t$ , the solar-system bound on  $\beta_4 \phi_0^4$  is similar to Eq. (5.18), i.e.,  $|\beta_4| \phi_0^4 \lesssim 10^{-5} M_{\text{pl}}^2$ .

For  $n \neq -2$ , the contribution of the coupling  $d_4$  survives in  $\gamma$ . At  $r = r_t$ , Eq. (5.45) reduces to

$$\gamma \simeq 1 - \frac{3\beta_4 \phi_0^4}{M_{\text{pl}}^2} - \frac{3 \cdot 2^{-n-1} (n+2) d_4 \phi_0^{2n+4}}{M_{\text{pl}}^2}. \quad (5.46)$$

The second term on the r.h.s. of Eq. (5.46) is similar to that for  $n = -2$ , so we obtain the bound  $|\beta_4| \phi_0^4 \lesssim 10^{-5} M_{\text{pl}}^2$ . Under the condition (5.27), the third term on the r.h.s. of Eq. (5.46) is compatible with the solar-system constraint for  $\phi_0 \lesssim 10^{-3} M_{\text{pl}}$ .

At the distance  $r_t < r < r_v$ , we substitute the solutions (5.40) into Eqs. (2.13) and (2.14). Then, we obtain the integrated solution

$$\Phi(r) \simeq \frac{\Phi_* r_*}{6r} \left[ 1 + \frac{\phi_0^2 \Phi_*}{6M_{\text{pl}}^2 x} - \frac{2^{n/2+3} \beta_4 \phi_0^4 \sqrt{\epsilon_0} x^{3/2}}{M_{\text{pl}}^2} - \frac{2^{-n} \phi_0^{2n+4} d_4}{M_{\text{pl}}^2} \left\{ 2^{n/2+3} (n+2) \sqrt{\epsilon_0} x^{3/2} + \frac{(10n+31)\Phi_*}{24x} \right\} \right], \quad (5.47)$$

$$\Psi(r) \simeq -\frac{\Phi_* r_*}{6r} \left[ 1 + \frac{\phi_0^2 \Phi_*}{6M_{\text{pl}}^2 x} + \frac{\beta_4 \phi_0^4 (2^{n/2+5} \sqrt{\epsilon_0} x^{3/2} + 3)}{2M_{\text{pl}}^2} + \frac{2^{-n} \phi_0^{2n+4} d_4}{M_{\text{pl}}^2} \left\{ 2^{n/2+4} (n+2) \sqrt{\epsilon_0} x^{3/2} - \frac{(8n+27)\Phi_*}{24x} \right\} \right]. \quad (5.48)$$

The post-Newtonian parameter is given by

$$\gamma \simeq 1 - \frac{3\beta_4 \phi_0^4 (2^{n/2+4} \sqrt{\epsilon_0} x^{3/2} + 1)}{2M_{\text{pl}}^2} - \frac{2^{-n-2} (n+2) \phi_0^{2n+4} d_4}{3M_{\text{pl}}^2} \left( 9 \cdot 2^{n/2+5} \sqrt{\epsilon_0} x^{3/2} + \frac{\Phi_*}{x} \right). \quad (5.49)$$

For  $n = -2$ , even though there are terms containing  $d_4$  in the gravitational potentials (5.47) and (5.48), the effect of the coupling  $d_4$  on  $\gamma$  disappears. As in the case  $d_4 = 0$  studied in Ref. [41], the local gravity constraint is satisfied for  $\phi_0 \lesssim 10^{-3} M_{\text{pl}}$ .

For  $n \neq -2$ , the contribution from the coupling  $d_4$  remains in  $\gamma$ . At  $r = r_v$ , the dimensionless distance  $x_v = r_v/r_*$  satisfies  $\sqrt{\epsilon_0} x_v^{3/2} = [2^{n/2+3} \phi_0^2 \{\beta_4 + 2^{-n} d_4 (n+2) \phi_0^{2n}\}]^{-1}$ . Substituting this relation into Eq. (5.49), ignoring the last term  $\Phi_*/x$  in Eq. (5.49), and taking the limit that the coupling  $d_4$  dominates over  $\beta_4$  such that  $|2^{-n} d_4 (n+2) \phi_0^{2n}| \gg |\beta_4|$ , the post-Newtonian parameter simply reduces to  $\gamma \simeq 1 - 3\phi_0^2/M_{\text{pl}}^2$ . Hence the resulting bound on  $\phi_0$  is the same as that for  $n = -2$ .

## VI. CONCLUSIONS

The beyond-generalized Proca theories were constructed in such a way that they recover the shift-symmetric GLPV theories in the scalar limit. In GLPV theories, solid angle deficit singularities arise in the case where the deviation from Horndeski theories (weighed by the parameter  $\alpha_H$ ) does not vanish at the center of a spherically symmetric body ( $r = 0$ ). The appearance of solid angle deficit singularities is associated with the fact that the gravitational potential  $\Phi(r)$  contains the non-vanishing constant  $-(1/2)\ln(1 + \alpha_H)$  in the limit  $r \rightarrow 0$ , see Eq. (3.15).

In this paper, we derived spherically symmetric solutions around the center of the compact body in quartic-order beyond-generalized Proca theories. For the model described by the functions (4.6) the resulting gravitational potentials around  $r = 0$  are given by Eqs. (4.10)-(4.11), so they satisfy the conditions (4.1)-(4.2) for the absence of solid angle deficit singularities. In fact, we showed that solid angle deficit singularities are generally absent in quartic-order beyond-generalized Proca theories with arbitrary functions  $G_{2,3,4}, g_4, f_4$ . This is mainly attributed to the fact that Eq. (4.15) contains the term  $2\Phi'/r - (1 - e^{2\Phi})/r^2$ , which demands the two conditions (4.17) for the consistency of Eq. (4.15). Existence of the temporal component  $\phi$  in beyond-generalized Proca theories leads to the vanishing constant  $\Phi_0$  in the expansion of  $\Phi$  around  $r = 0$ . This is not the case for GLPV theories in which the extra term  $1 + \alpha_H$  in the square bracket of Eq. (3.11) gives rise to the non-vanishing constant  $\Phi_0$ .

We also studied the Vainshtein mechanism in the presence of the quartic-order beyond-generalized Proca Lagrangian with the coupling  $f_4(X) = d_4 X^n$  besides vector-Galileon terms. If the cubic vector-Galileon term is absent ( $\beta_3 = 0$ ), we obtained the solution where the derivative  $\chi'$  of the longitudinal scalar exactly vanishes. In this case, the beyond-generalized Proca interaction does not give rise to any contribution to the temporal vector component  $\phi$ , whose solution outside the body ( $r > r_*$ ) is given by Eq. (5.12). Provided that the condition (5.18) is satisfied, the corrections to the gravitational potentials  $\Phi$  and  $\Psi$  induced by the vector field are sufficiently small such that the theories are compatible with solar-system constraints.

We also derived the vector-field profiles in the case where the coupling  $\beta_3$  is present besides  $\beta_4$  and  $d_4$ . The radial dependence of the vector field is similar to that for  $d_4 = 0$  apart from the difference of some coefficients, so the Vainshtein mechanism works in a similar way to that discussed in Ref. [41]. We also found that there are corrections to the leading-order gravitational potentials  $\Phi$  and  $\Psi$  from the new coupling  $d_4$ , but as long as the coupling  $d_4$  is in the range with  $\phi_0 \lesssim 10^{-3} M_{\text{pl}}$ , the solar-system bound on  $\gamma = -\Phi/\Psi$  is well satisfied.

We have thus shown that the quartic-order beyond-generalized Proca theories have a nice feature for avoiding the problem of solid angle deficit singularities at the center of the compact body, while allowing the success of the Vainshtein screening outside the body. Since the appearance of solid angle deficit singularities in GLPV theories is intrinsically related to the beyond-Horndeski Lagrangian at quartic order [22], we anticipate that the absence of solid angle deficit singularities beyond-generalized Proca theories should persist in more general cases containing the fifth- and sixth-order Lagrangians derived in Ref. [44]. Nevertheless this requires further detailed study, so we leave the analysis for the derivation of spherically symmetric solutions in full beyond-generalized Proca theories as a future work.

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### Appendix A: Coefficients

The coefficients in Eqs. (2.13)-(2.17) are given by

$$\begin{aligned}
C_1 &= 4X_\phi(1 - 2g_4 - 4X_\phi g_{4,X}), \quad C_2 = 4X_\phi \chi' G_{3,X} + 2\phi\phi' e^{2\Psi}(1 - 2g_4 - 4X_\phi g_{4,X}), \\
C_3 &= -32X_\phi X_\chi [G_{4,XX} + 3f_4 + 2(X_\phi + X_\chi)f_{4,X}], \quad C_4 = -2G_{3,X}(X_\phi + X_\chi)\chi', \\
C_5 &= -4G_4 + 8(X_\phi + 2X_\chi)G_{4,X} + 16X_\chi(X_\phi + X_\chi)G_{4,XX} + 16X_\chi(7X_\phi + 5X_\chi)f_4 + 32X_\chi(X_\phi + X_\chi)^2 f_{4,X}, \\
C_6 &= -e^{2\Phi}(G_2 - 2X_\phi G_{2,X}) + [\chi'\phi\phi' e^{2\Psi} + 2(X_\phi + X_\chi)\chi''] G_{3,X} + e^{2\Psi}\phi'^2(1 - 2g_4 - 4X_\phi g_{4,X})/2, \\
C_7 &= 4X_\phi \chi' G_{3,X} + 4[G_{4,X} + 2(X_\phi + X_\chi)G_{4,XX} + 2(5X_\phi + 4X_\chi)f_4 + 4(X_\phi + X_\chi)^2 f_{4,X}] e^{-2\Phi} \chi' \chi'' \\
&\quad - 8X_\chi [G_{4,XX} + 3f_4 + 2(X_\phi + X_\chi)f_{4,X}] e^{2\Psi} \phi\phi', \\
C_8 &= 2(1 - e^{2\Phi})(G_4 - 2X_\phi G_{4,X}) - 4X_\chi G_{4,X} - 8X_\phi X_\chi G_{4,XX} \\
&\quad - 8X_\chi(5X_\phi + X_\chi)f_4 - 16X_\phi X_\chi(X_\phi + X_\chi)f_{4,X}, \\
C_9 &= 4X_\phi(1 - 2g_4 - 4X_\chi g_{4,X}), \quad C_{10} = -2G_{3,X}(X_\phi - X_\chi)\chi' + 2\phi\phi' e^{2\Psi}(1 - 2g_4 - 4X_\chi g_{4,X}), \\
C_{11} &= 4G_4 + 8(X_\phi - 2X_\chi)G_{4,X} + 16X_\chi(X_\phi - X_\chi)G_{4,XX} + 16X_\chi(3X_\phi - 5X_\chi)f_4 + 32X_\chi(X_\phi^2 - X_\chi^2)f_{4,X}, \\
C_{12} &= -e^{2\Phi}(G_2 - 2G_{2,X}X_\chi) - e^{2\Psi}G_{3,X}\chi'\phi\phi' + \phi'^2 e^{2\Psi}(1 - 2g_4 - 4X_\chi g_{4,X})/2, \\
C_{13} &= 4G_{3,X}\chi'X_\chi + 8e^{-2\Phi}X_\phi\chi'\chi''f_4 + 4[2X_\chi G_{4,XX} + 10f_4X_\chi + G_{4,X} + 4X_\chi(X_\phi + X_\chi)f_{4,X}] e^{2\Psi}\phi\phi', \\
C_{14} &= 2(1 - e^{2\Phi})G_4 - 4(2 - e^{2\Phi})X_\chi G_{4,X} - 8G_{4,XX}X_\chi^2 - 8X_\chi(X_\phi + 5X_\chi)f_4 - 16X_\chi^2(X_\phi + X_\chi)f_{4,X}, \\
C_{15} &= 2G_4 - 8f_4X_\chi^2 + 4(X_\phi - X_\chi)G_{4,X}, \quad C_{16} = 8f_4X_\phi X_\chi, \\
C_{17} &= -4X_\phi(1 - 2g_4) + 2G_4 + 4(4X_\phi - X_\chi)G_{4,X} + 8X_\phi(X_\phi - X_\chi)G_{4,XX} - 8X_\chi^2(f_4 + 2X_\phi f_{4,X}), \\
C_{18} &= -2G_4 + 8X_\chi(3X_\phi + 5X_\chi)f_4 + 16X_\chi(X_\phi^2 + X_\chi^2)f_{4,X} - 4(X_\phi - 2X_\chi)G_{4,X} - 8X_\chi(X_\phi - X_\chi)G_{4,XX}, \\
C_{19} &= -8X_\chi X_\phi(3f_4 + 2X_\chi f_{4,X}), \\
C_{20} &= 2X_\phi \chi' G_{3,X} - 2\phi e^{2\Psi}(1 - 2g_4)\phi' \\
&\quad + 4[3G_{4,X} + (2X_\phi - X_\chi)G_{4,XX} + 3f_4X_\chi + 2X_\chi(X_\phi - X_\chi)f_{4,X}] \phi\phi' e^{2\Psi} \\
&\quad + 2[G_{4,X} - 2(X_\phi - X_\chi)G_{4,XX} + 2(3X_\phi + 4X_\chi)f_4 + 4(X_\phi^2 + X_\chi^2)f_{4,X}] e^{-2\Phi} \chi' \chi'', \\
C_{21} &= 2G_4 + 4(X_\phi - X_\chi)G_{4,X} - 8X_\phi X_\chi G_{4,XX} - 8X_\chi(3X_\phi + X_\chi)f_4 - 16X_\phi X_\chi(X_\phi + X_\chi)f_{4,X}, \\
C_{22} &= -2G_{3,X}\chi'X_\chi - 2[G_{4,X} + 2X_\chi G_{4,XX} + 2f_4X_\chi - 4X_\chi(X_\phi - X_\chi)f_{4,X}] \phi\phi' e^{2\Psi} \\
&\quad - 4e^{-2\Phi}X_\phi(5f_4 + 4X_\chi f_{4,X})\chi' \chi'', \\
C_{23} &= -2G_4 + 8X_\chi(G_{4,X} + X_\chi G_{4,XX}) + 8X_\chi(3X_\phi + 5X_\chi)f_4 + 16X_\chi^2(X_\phi + X_\chi)f_{4,X}, \\
C_{24} &= \left[ G_{3,X}\chi'\phi\phi' + 2(G_{4,X} + 2f_4X_\chi)\phi\phi'' + 2(G_{4,X} + 2G_{4,XX}X_\phi + 2X_\chi f_4 + 4X_\chi X_\phi f_{4,X})\phi'^2 \right] e^{2\Psi} \\
&\quad - \left[ G_2 + (1 - 2g_4)e^{2\Psi-2\Phi}\phi'^2/2 \right] e^{2\Phi} + 2[X_\chi G_{3,X} - \{G_{4,XX} - 2(X_\phi - X_\chi)f_{4,X}\} e^{2\Psi-2\Phi}\phi\phi'\chi'] \chi'' \\
&\quad + 4X_\phi e^{-2\Phi} \left[ (f_4 + 2X_\chi f_{4,X})\chi''^2 + f_4\chi'\chi''' \right], \\
C_{25} &= 2[G_{4,X} - 2X_\chi G_{4,XX} - 4X_\chi f_4 - 4X_\chi(X_\phi + X_\chi)f_{4,X}] \phi\phi' e^{2\Psi} \\
&\quad + 2[G_{4,X} + 2X_\chi G_{4,XX} + 4(X_\phi + 2X_\chi)f_4 + 4X_\chi(X_\phi + X_\chi)f_{4,X}] e^{-2\Phi} \chi' \chi'', \\
D_1 &= -2(1 - 2g_4)\phi, \quad D_2 = 2\phi(1 - 2g_4 - 4X_\chi g_{4,X}), \\
D_3 &= G_{3,X}\chi'\phi - 4e^{-2\Phi}\phi g_{4,X}\chi'\chi'' - (3 - 6g_4 - 4X_\phi g_{4,X})\phi', \\
D_4 &= -4[1 - 2g_4 + 2X_\chi G_{4,XX} + 6X_\chi f_4 + 4X_\chi(X_\phi + X_\chi)f_{4,X}]\phi, \\
D_5 &= (1 - 2g_4 - 4X_\chi g_{4,X})\phi' - G_{3,X}\chi'\phi, \\
D_6 &= 4[G_{4,X} + 2X_\chi G_{4,XX} + 10f_4X_\chi + 4X_\chi(X_\phi + X_\chi)f_{4,X}]\phi, \\
D_7 &= (G_{2,X}e^{2\Phi} + G_{3,X}\chi'' + e^{2\Psi}g_{4,X}\phi'^2)\phi - (1 - 2g_4)\phi'' - 2e^{-2\Phi}g_{4,X}\phi'\chi'\chi'', \\
D_8 &= 2[G_{3,X}\chi' + 2\{G_{4,XX} + 4f_4 + 2(X_\phi + X_\chi)f_{4,X}\}\chi'\chi''e^{-2\Phi}]\phi - 2(1 - 2g_4)\phi', \\
D_9 &= 2[(e^{2\Phi} - 1)G_{4,X} - 2X_\chi G_{4,XX} - 8X_\chi f_4 - 4X_\chi(X_\phi + X_\chi)f_{4,X}]\phi, \\
D_{10} &= 8X_\phi g_{4,X}\chi', \quad D_{11} = 4g_{4,X}e^{2\Psi}\phi\phi'\chi' - 2(X_\phi - X_\chi)G_{3,X}e^{2\Phi}, \\
D_{12} &= 4[G_{4,X} - 2(X_\phi - X_\chi)G_{4,XX} - 2(3X_\phi - 4X_\chi)f_4 - 4(X_\phi^2 - X_\chi^2)f_{4,X}]\chi', \\
D_{13} &= 8\chi'X_\phi f_4, \quad D_{14} = (g_{4,X}e^{2\Psi}\phi'^2 - G_{2,X}e^{2\Phi})\chi' - G_{3,X}e^{2\Psi+2\Phi}\phi'\phi, \\
D_{15} &= 4X_\chi G_{3,X}e^{2\Phi} - 4[G_{4,XX} + 4f_4 + 2(X_\phi + X_\chi)f_{4,X}] e^{2\Psi}\phi\phi'\chi', \\
D_{16} &= 2[(1 - e^{2\Phi})G_{4,X} + 2X_\chi G_{4,XX} + 8X_\chi f_4 + 4X_\chi(X_\phi + X_\chi)f_{4,X}]\chi'. \tag{A1}
\end{aligned}$$



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